Anomalous scaling of a triple correlation function of a randomly advected passive scalar

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For a short-correlated Gaussian velocity field the problem of the passive scalar with the imposed constant gradient is considered. It is shown that the scaling of the three-point correlation function is anomalous. In the limit of large dimension of space d, the anomalous exponent is calculated. [S1063-651X(96)02610-4]

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We consider the problem of a passive scalar Θ advected by δ correlated in time velocity field $\boldsymbol{v}$ [1]. The scalar obeys an equation

$$\partial_t\Theta+\boldsymbol{v}\cdot\nabla\Theta=\kappa\Delta\Theta. \tag{1}$$

Here κ is the molecular diffusivity, which determines the behavior of the passive scalar for small distances. The velocity $\boldsymbol{v}$ is a random function to be defined below. There is no pumping of a passive scalar into the system, but instead a condition on the mean value of a passive scalar is imposed, namely, it has a constant gradient $\boldsymbol{g}$ in space,

$$\Theta=\boldsymbol{g}\cdot\boldsymbol{r}+\theta. \tag{2}$$

Here θ designate the fluctuating part of a scalar field

$$\langle\theta\rangle=0. \tag{3}$$

By $\langle\rangle$ we mean statistical average. This problem was previously considered by Shraiman and Siggia in [2], where they derived the equation for the n-point correlation function and found the pair correlation function. In [3,4] they proposed a phenomenological equation describing the n-point correlation functions of a passive scalar.

The description of this problem, analogous to Kolmogorov dimensional analysis, gives the correct answer for the two-point correlation function $f(r_{12})=\langle\theta(r_1)\theta(r_2)\rangle$. However, it has a deviation from experimental data [5,6] for higher-order correlation functions. We will concentrate on such a deviation for the three-point correlation function $\Gamma=\langle\theta(r_1)\theta(r_2)\theta(r_3)\rangle$, which is the lowest that has anomalous behavior (i.e., different from the one obtained by a dimensional estimate). The imposed gradient of the mean value looks like breaking of translational and rotational invariance. However, the correlation functions of θ are translational invariant and the gradient of the mean value of a scalar field will effectively play the role of anisotropic pumping. There is a significant difference between this problem and the problem with isotropic pumping. Due to the chosen direction in space, the odd order correlation functions do not vanish.

We assume the velocity field to be Gaussian and δ correlated in time [1]. Therefore the statistics of $\boldsymbol{v}$ is completely determined by the pair-correlation function

$$\langle v^\alpha(r_1,t_1)v^\beta(r_2,t_2)\rangle=V^{\alpha\beta}(r_{12})\delta(t_1-t_2), \tag{4}$$

$$V^{\alpha\beta}(r)=V_0\delta^{\alpha\beta}-K^{\alpha\beta}(r). \tag{5}$$

Here V_0 is a large constant having the meaning of average value of $\boldsymbol{v}^2$, $K^{\alpha\beta}$ is the eddy diffusivity tensor

$$K^{\alpha\beta}(r)=Dr^{-\gamma}\left[\frac{d+1-\gamma}{2-\gamma}r^2\delta^{\alpha\beta}-r^\alpha r^\beta\right], \tag{6}$$

where d is the dimension of space, and γ is parameter that determines the scaling of the velocity pair-correlation function. The parameter γ is supposed to be between zero and 2. The powerlike behavior (6) is valid up to some $r\sim L$, where L is the correlation length of velocity. For $r>L$ the correlator $V^{\alpha\beta}$ decreases and goes to zero as $r\to\infty$. This means that $K^{\alpha\beta}(r)=V_0\delta^{\alpha\beta}$ for $r\gg L$. As we will see later, the precise form of this decrease is not important. The relation between constants is

$$L^{2-\gamma}=\frac{2-\gamma}{D(d+1-\gamma)}V_0. \tag{7}$$

We assume that the cutoff length L is d independent, so that V_0 explicitly depends on space dimensionality.

Our aim is to find the scaling exponent of the triple correlation function deep inside convective interval, that is, for $r_d\ll r\ll L$, where r_d is a diffusion scale

$$r_d^{2-\gamma}=\frac{2\kappa(2-\gamma)}{D(d-1)}. \tag{8}$$

The condition $r\gg r_d$ implies that we can disregard diffusion [8].

From (1) one can derive the equation for the nth-order correlation function [1,2]

$$\hat{\mathcal{L}}\langle\Theta_1\cdots\Theta_n\rangle=0. \tag{9}$$

Here the operator $\hat{\mathcal{L}}$ contains operators of both turbulent and molecular diffusion

$$\hat{\mathcal{L}}=\tfrac{1}{2}\sum_{i,j}V^{\alpha\beta}(r_{ij})\nabla_i^\alpha\nabla_j^\beta+\kappa\sum_i\nabla_i^2. \tag{10}$$

We will see that the scaling of the pair-correlation function is normal. For the three-point correlation function the presence of zero modes of the operator $\hat{\mathcal{L}}$ makes the scaling anomalous, which is different from the $1+\gamma$ obtained from

Kolmogorov-like estimates. The exception is the special case when the scaling exponent of the velocity correlation function γ is zero [2]. Note also that for the scaling exponent equal to 2 the operator $\hat{\mathcal{L}}$ has a singularity and special treatment is needed [7]. Following [8], we will calculate the anomalous scaling exponent in the limit of large d. For $d\to\infty$ the problem can be solved exactly. Then, finding corrections in the next order over $1/d$ one can calculate the anomalous exponent of the three-point correlator.

To calculate the triple correlation function one should know the two-point correlator since it explicitly enters the equation. The equation for the two-point correlation function is

$$\hat{\mathcal{L}}^{(p)}f(\boldsymbol{r}_{12})=-\hat{\mathcal{L}}[(\boldsymbol{g}\cdot\boldsymbol{r}_1)(\boldsymbol{g}\cdot\boldsymbol{r}_2)]. \tag{11}$$

The operator $\hat{\mathcal{L}}^{(p)}$ may be written as

$$\hat{\mathcal{L}}^{(p)}=\frac{D(d-1)}{2-\gamma}r^{1-d}\partial_r(r^{d+1-\gamma}\partial_r),\quad r_d\ll r<L. \tag{12}$$

We should match the solution obtained at $r<L$ with the solution at $r>L$. For this region the operator has the form

$$\hat{\mathcal{L}}^{(p)}=V_0r^{1-d}\partial_r(r^{d-1}\partial_r). \tag{13}$$

Since we will consider the three-point correlation function at large d, we calculate the pair-correlation function in the same limit (it can be found exactly as well). The solution of Eq. (11) at $r<L$ and $d\gg1$ that satisfies the boundary condition at $r\sim L$ is

$$f(r)=\frac{g^2L^2}{d}\left[\frac{2-\gamma}{2\gamma}-\frac{1}{\gamma}\left(\frac{r}{L}\right)^{\gamma}+\frac{1}{2}\left(\frac{r}{L}\right)^{2}\right]. \tag{14}$$

At $r>L$ the pair correlation function is zero in the main order of the $1/d$ expansion. For r deep inside the convective interval the r^2 term is small in comparison to r^{γ}. Therefore we see that the scaling behavior of a two-point correlation function is normal, as shown by Shraiman and Siggia [3]. Note that (14) is isotropic in the convective interval. The anisotropic part has a scaling exponent larger than γ and therefore in the convective interval it is much smaller. Besides, it has additional factor $1/d$ with respect to the isotropic part.

For the three-point correlation function one can derive the equation

$$\hat{\mathcal{L}}\Gamma=\Phi_{12,3}+\Phi_{13,2}+\Phi_{23,1}, \tag{15}$$

where

$$\Phi_{12,3}=\nabla_1^{\alpha}f(r_{12})g^{\beta}[K^{\alpha\beta}(r_{13})-K^{\alpha\beta}(r_{23})]. \tag{16}$$

Since the preferable direction exists in the problem, then Γ depends on five independent variables in any dimension of space $d>2$. It is convenient to choose the set

$$\Gamma=\Gamma(r_{12},r_{13},r_{23},\boldsymbol{r}_{12}\cdot\boldsymbol{g},\boldsymbol{r}_{23}\cdot\boldsymbol{g}). \tag{17}$$

However, it is impossible to find symmetric parametrization because of the identity $\boldsymbol{r}_{12}+\boldsymbol{r}_{13}+\boldsymbol{r}_{23}=0$.

Unfortunately, the number of variables is too large to solve Eq. (15) exactly at arbitrary γ and d. There are several cases when the problem can be solved exactly: the cases of $\gamma=0$ [9,10,3], large d [8], and $\gamma=2$ [11]. After solving the problem exactly at these special values of parameters one can develop the perturbation theory for the small deviations from these values. In the cases of large d and $\gamma=2$ this perturbation theory can be performed. The case of $\gamma=0$ corresponds to the strongly degenerate perturbation theory. We will solve Eq. (15) in the limit of large d, where perturbation theory is regular. To do this one should keep in the operator (10) only terms that are the largest at large d and solve the equation. Then one can look for a correction due to a finite value of the parameter $1/d$. Justification of this procedure can be found in [8].

In variables (17) the main part of the operator $\hat{\mathcal{L}}$ can be written as

$$\hat{\mathcal{L}}_0=\frac{d^2D}{2-\gamma}\sum_{i<j}r_{ij}^{1-\gamma}\partial_{r_{ij}}. \tag{18}$$

For large d and $r\ll L$ one can substitute (14) into the right-hand side of (15). The result is

$$\Phi_{12,3}^{(0)}=-\frac{g^2DL^{2-\gamma}}{2-\gamma}[(\boldsymbol{r}_{12}\cdot\boldsymbol{g})r_{12}^{\gamma-2}(r_{13}^{2-\gamma}-r_{23}^{2-\gamma})]. \tag{19}$$

Thus, in the first step of the iteration procedure we should solve the equation

$$\hat{\mathcal{L}}_0\Gamma_0=\Phi_{12,3}^{(0)}+\Phi_{13,2}^{(0)}+\Phi_{23,1}^{(0)}. \tag{20}$$

The solution can be found by integrating over the characteristics of the operator (18). An important question is that of boundary conditions. We should supply Eq. (20) with the boundary conditions at zero and infinity. However, this equation is of first order and only one condition can be satisfied. Since we omitted the diffusion part of the operator (10), we should pose the only condition $\Gamma\to0$ as $r\to\infty$. One can show [8] that solution obtained in this way can be matched with the diffusion region.

One can write the following solution of (20), which satisfies the above boundary condition:

$$\Gamma_0=-\frac{2-\gamma}{\gamma Dd^2}\int_0^{\infty}dt\Phi_{12,3}^{(0)}[\tilde{r}_{12},\tilde{r}_{13},\tilde{r}_{23},(\boldsymbol{r}_{12}\cdot\boldsymbol{g}),(\boldsymbol{r}_{13}\cdot\boldsymbol{g})] + \text{(permutations)}, \tag{21}$$

where

$$\tilde{r}_{ij}(t)=(r_{ij}^{\gamma}+t)^{1/\gamma}. \tag{22}$$

The main contribution to Γ_0 comes from the region $t\lesssim L^{\gamma}$. Therefore we can use the expression (19) for $\Phi_{12,3}^{(0)}$ and write L^{γ} as the upper limit of the integration. The precise form of this cutoff is not important since the integral behaves logarithmically on the upper limit. Finally, we get

$$\Gamma_0=\frac{g^2L^{2-\gamma}}{\gamma d^2}(\boldsymbol{r}_{12}\cdot\mathbf{g})\int_0^{L^\gamma}dt(r_{12}^\gamma+t)^{1-2/\gamma}\times[(t+r_{13}^\gamma)^{2/\gamma-1}-(t+r_{23}^\gamma)^{2/\gamma-1}]+\text{ permutations.}\tag{23}$$

The integral (23) cannot be calculated analytically for arbitrary γ. However, we are interested in Γ in the convective interval, that is, for $r\ll L$. There main contribution to (23) comes from large t. Expanding the integrand over r/L, we find the following result in the main logarithmic order:

$$\Gamma_0=\frac{2(2-\gamma)}{d}\ln\left(\frac{L}{r}\right)Z_0,\tag{24}$$

where r in the logarithm is of order r_{ij} and

$$Z_0=\frac{g^2L^{2-\gamma}}{2\gamma d}[(\boldsymbol{r}_{12}\cdot\mathbf{g})(r_{13}^\gamma-r_{23}^\gamma)+(\boldsymbol{r}_{13}\cdot\boldsymbol{g})(r_{12}^\gamma-r_{23}^\gamma)+(\boldsymbol{r}_{23}\cdot\boldsymbol{g})(r_{12}^\gamma-r_{13}^\gamma)]\tag{25}$$

Note that Z_0 is necessarily a zero mode of the operator (18). It is also worth mentioning that Z_0 scales as $\gamma+1$ and this is the only zero mode of the operator $\hat{\mathcal{L}}_0$ with such scaling.

Now we should find a correction to (24) in the next order over $1/d$. To do this we should solve the equation

$$\hat{\mathcal{L}}_0\Gamma_1=-\hat{\mathcal{L}}_1\Gamma_0.\tag{26}$$

Here $\hat{\mathcal{L}}_1$ is the part of operator (10) that is proportional to d. In Eq. (26) we disregard terms that come from the right-hand side of Eq. (15) since they do not contain the logarithm. Solving Eq. (26), we find for the main logarithmic term

$$\Gamma_1=\frac{2(2-\gamma)^2}{d^2}Z_0\ln^2\left(\frac{L}{r}\right).\tag{27}$$

We can continue this iteration procedure and find the main logarithmic subsequence. As a result, we obtain the series

$$\Gamma=Z_0\left[1+\Delta\ln\left(\frac{L}{r}\right)+\frac{\Delta^2}{2}\ln^2\left(\frac{L}{r}\right)+\cdots\right]-Z_0=Z_0\left(\frac{L}{r}\right)^\Delta-Z_0,\tag{28}$$

where the anomalous exponent Δ that makes a deviation from normal scaling is

$$\Delta=\frac{2(2-\gamma)}{d}.\tag{29}$$

We see that the solution consists of two parts. The second term on the right-hand side of (28) has normal scaling $\gamma+1$. One can check that it is a partial solution of the inhomogeneous equation (15). However, if it were alone, the solution would not satisfy boundary conditions. To ensure it we have the first term in (28). One can see that this contribution gives anomalous scaling with the exponent $\gamma+1-\Delta$. This term is the solution of the homogeneous equation $\hat{\mathcal{L}}\Gamma=0$ and therefore is the zero mode of the operator $\hat{\mathcal{L}}$ [8,11,3].

To conclude, we have shown that the three-point correlation function has the scaling exponent $\gamma+1-\Delta$, which differs from naive dimensional estimates. This exponent was analytically calculated in the leading $1/d$ order.

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